

KOLMOGOROV COMPLEXITY AND GENERALIZED LENGTH FUNCTIONS

CAMERON FRAIZE AND CHRISTOPHER P. PORTER

ABSTRACT. Kolmogorov complexity measures the algorithmic complexity of a finite binary string σ in terms of the length of the shortest description σ^* of σ . Traditionally, the length of a string is taken to measure the amount of information contained in the string. However, we may also view the length of σ as a measure of the cost of producing σ , which permits one to generalize the notion of length, wherein the cost of producing a 0 or a 1 can vary in some prescribed manner.

In this article, we initiate the study of this generalization of length based on the above information cost interpretation. We also modify the definition of Kolmogorov complexity to use such generalized length functions instead of standard length. We further investigate conditions under which the notion of complexity defined in terms of a given generalized length function preserves some essential properties of Kolmogorov complexity. We focus on a specific class of generalized length functions that are intimately related to a specific subcollection of Bernoulli p -measures, namely those corresponding to the unique computable real $p \in (0, 1)$ such that $p^k = 1 - p$, for integers $k \geq 1$. We then study randomness with respect to such measures, by proving a generalization version of the classic Levin-Schnorr theorem that involves k -length functions and then proving subsequent results that involve effective dimension and entropy.

1. INTRODUCTION

Kolmogorov complexity provides a measure of algorithmic complexity for finite binary strings in terms of quantity of information, expressed by the standard length function for strings, hereafter denoted $|\cdot|$. If instead we view the length of σ as a measure of the cost of producing σ , which permits one to generalize the notion of length, wherein the cost of producing a 0 or a 1 can vary in some computable way. The goal of this paper is to inaugurate the study of such generalized length functions in the context of Kolmogorov complexity. We will focus in particular on what we refer to as *k-length functions*, which define the cost of producing a 0 to be one bit and the cost of producing a 1 to be k bits. As we will see, generalizing Kolmogorov complexity to such length functions yields a notion of complexity that behaves much like Kolmogorov complexity (satisfying analogues of many of the properties of standard Kolmogorov complexity) but is particularly useful in characterizing Martin-Löf randomness with respect to a specific kind of Bernoulli measure along the lines of the classic Levin-Schnorr theorem.

The contents of this paper are as follows. In Section 2, we introduce the requisite background in computability theory and algorithmic randomness. In Section 3, we introduce the concept of a *generalized length function* ℓ and define a variant of Kolmogorov complexity by replacing the standard notion of the length of a description with that of the generalized length of description for a binary string σ . For generalized length functions ℓ , we call this ℓ -Kolmogorov complexity, denoted $K^{(\ell)}$. Lastly, we identify a certain natural class of generalized length functions ℓ that allow $K^{(\ell)}$ to preserve basic properties of prefix-free Kolmogorov complexity.

In addition we isolate a subclass of the Bernoulli p -measures, referred to as *p_k -measures*, which are Bernoulli measures given by a parameter p satisfying $p^k = 1 - p$ for a fixed $k \geq 1$. We denote these measures by λ_k for every such k . We show that p_k -measures are intimately connected to k -length functions, we investigate the sequence of values given by the number of strings of ℓ_k -length n for a fixed $k \geq 1$, and uncover a Fibonacci sequence-like structure for every such sequence. Last, we provide of a generalization of the classic KC-theorem for k -length,

the proof of which yields a significant simplification of the proof of the original version of the theorem.

In Section 4, we study possible generalizations of the classic Levin-Schnorr theorem in terms of generalized length functions. For $j, k \geq 1$, we arrive at a characterization of λ_j -Martin-Löf random sequences in terms of ℓ_k -Kolmogorov complexity of their initial segments, modulo a multiplicative constant that accounts for the differences between k -length and j -length. We also show that the above-mentioned multiplicative constant in our generalization of the Levin-Schnorr theorem is necessary by showing that there are sequences X such that (i) the initial segments of X have $K^{(k)}$ -complexity above a threshold that does not include the multiplicative constant but (ii) are not random with respect to any computable measure.

Lastly, in Section 5, we modify the notions of effective packing dimension and effective Hausdorff dimension using $K^{(k)}$ and provide a partial generalization of a result of Hoyrup's involving the relationship between effective dimension and entropy for randomness with respect to any Bernoulli p_k -measure.

We fix the following notation and terminology. We denote the set of natural numbers by ω and the set of finite binary strings by $2^{<\omega}$, with ϵ denoting the empty string. We shall use lowercase letters such as n, m to denote natural numbers, and lowercase Greek letters such as σ and τ to denote binary strings. All logarithms without subscripts (i.e. $\log x$ for $x > 0$) will be base 2, unless otherwise stated. We use 2^ω to denote *Cantor space*, set of infinite binary sequences, and use capital letters such as X and Y to denote such sequences. The set of non-negative dyadic rationals, of the form $m/2^n$ for $m, n \in \omega$ is denoted \mathbb{Q}_2 . If $X \in 2^\omega$ and $n \in \omega$, then $X \upharpoonright n$ is the first n bits of X , and $X(n)$ is the $(n+1)^{\text{st}}$ bit of X . If σ and τ are binary strings, then $\sigma \preceq \tau$ means that σ is an initial segment of τ , i.e. $\tau \upharpoonright |\sigma| = \sigma$. Similarly, for $X \in 2^\omega$, $\sigma \prec X$ means that σ is an initial segment of X , and the *cylinder set* $[\![\sigma]\!]$ is the set of all $X \in 2^\omega$ such that $\sigma \prec X$. For strings σ and τ , $\sigma \frown \tau$ is the concatenation of σ and τ . Given σ , $\#_0(\sigma)$ is the number of 0's in σ , and $\#_1(\sigma)$ is the number of 1's in σ . If $S \subseteq 2^{<\omega}$, then we let $[\![S]\!]$ denote $\bigcup_{\sigma \in S} [\![\sigma]\!]$. The cylinder sets form a basis for the usual topology on Cantor space (the product topology), and so the open sets in 2^ω are of the form $[\![S]\!]$ for $S \subseteq 2^{<\omega}$. An open set U is said to be *effectively open* (or Σ_1^0) if there is a computably enumerable (hereafter, c.e.) set $S \subseteq 2^{<\omega}$ such that $U = [\![S]\!]$. A sequence $\{U_n\}_{n \in \omega}$ is said to be *uniformly* Σ_1^0 if there exists a sequence $\{S_n\}_{n \in \omega}$ of uniformly c.e. sets such that $U_n = [\![S_n]\!]$.

2. BACKGROUND

2.1. Kolmogorov complexity and prefix-free machines. We assume that the reader is familiar with the basics of computability theory. See for instance Soare [Soa87, Ch. I-IV], Nies [Nie09, Ch. 1], or Downey and Hirschfeldt [DH10, Ch. 2].

Definition 2.1. (i) For a Turing machine M , let $\text{dom}(M)$ be

$$\{\sigma : (\exists s)(M(\sigma)[s] \downarrow)\},$$

where $M(\sigma)[s] \downarrow$ means that there is some $t \leq s$ such that M halts on the string σ in t computational steps.

- (ii) If $f : 2^{<\omega} \rightarrow 2^{<\omega}$ is a partial function, then f is called *partial computable* if there is some Turing machine M such that $\text{dom}(f) = \text{dom}(M)$ and for all $\sigma \in \text{dom}(f)$, $M(\sigma) = f(\sigma)$. If f is total and some Turing machine computes f , then we simply call f a *computable function*.
- (iii) A *universal* Turing machine is a Turing machine U that simulates all other Turing machines; i.e. for every Turing machine M , there exists a string ρ_M , called the *coding constant* of M , such that $U(\rho_M \frown \sigma) = M(\sigma)$ for all $\sigma \in \text{dom}(M)$.

We will also be restricting our attention to a certain class of Turing computable functions, and for this we need a certain restriction on the domains of such functions.

Definition 2.2. A set $S \subseteq 2^{<\omega}$ is called *prefix-free* if for all $\sigma \in S$, there is no $\tau \in S$ such that $\sigma \prec \tau$, i.e., no τ extends σ .

Now we may define a notion of prefix-freeness for a Turing machine:

Definition 2.3. A Turing machine M is called a *prefix-free machine* if $\text{dom}(M)$ is a prefix-free set. We will refer to Turing machines that are not prefix-free as *plain machines*.

We now fix a specific prefix-free universal machine U : given a computable listing $\{M_e\}_{e \in \omega}$ of all prefix-free machines, let U be defined by $U(0^e 1 \sigma) = M_e(\sigma)$ for all $e \in \omega$ and $\sigma \in \text{dom}(M_e)$.

2.2. Kolmogorov complexity and information content measures. As mentioned in the introduction, *Kolmogorov complexity* measures the minimum amount of information needed to produce a given string σ from a Turing machine. Here we follow the standard presentations of Kolmogorov complexity such as Nies [Nie09] or Downey and Hirschfeldt [DH10].

Definition 2.4. Given a prefix-free machine M and a string $\sigma \in \text{dom}(M)$, the *prefix-free Kolmogorov complexity relative to M* , denoted $K_M(\sigma)$, of a string $\sigma \in 2^\omega$ is the length of the shortest input of M that outputs σ ; i.e.,

$$K_M(\sigma) = \min\{|\tau| : M(\tau) \downarrow = \sigma\}.$$

We can use universal machines to define a minimal Kolmogorov complexity as well:

Definition 2.5. The *prefix-free Kolmogorov complexity* $K(\sigma)$ of a string $\sigma \in 2^{<\omega}$ is the Kolmogorov complexity relative to a universal prefix-free machine U .

The key fact about prefix-free Kolmogorov complexity is that it is universal in the following sense:

Proposition 2.6 (Invariance property). *For every prefix-free machine M , there is a c such that for every σ ,*

$$K(\sigma) \leq K_M(\sigma) + c.$$

A number of results about K utilize the invariance property, and, indeed, we consider this property essential for Kolmogorov complexity. Thus, in the sequel we will require that generalizations of Kolmogorov complexity satisfy invariance.

A proof of the converse of the Levin-Schnorr theorem, which we will generalize in Section 4, utilizes the aforementioned KC theorem and a useful class of functions developed by Chaitin, called *information content measures*. When we generalize the KC theorem to a certain class of generalized length functions, we will generalize the concept of an information content measure for these functions as well. The KC theorem is as follows:

Theorem 2.7 (KC Theorem). *Let $\{(r_i, \tau_i)\}_{i \in \omega}$ be a computable sequence of pairs (called requests) with $r_i \in \omega$ and $\tau_i \in 2^{<\omega}$ for every i , such that $\sum_{i \in \omega} 2^{-r_i} \leq 1$. Then there exists a prefix-free machine M and sequence $\{\sigma_i\}_{i \in \omega}$ of strings with $|\sigma_i| = r_i$ such that $\text{dom}(M) = \{\sigma_i : i \in \omega\}$ and $M(\sigma_i) = \tau_i$ for every i . Furthermore, one can obtain an index for M effectively from an index of our sequence of requests.*

We call a set of requests $\{(r_i, \tau_i) : i \in \omega\}$ that satisfies the hypotheses of the KC theorem a *KC set*.

Definition 2.8. An *information content measure* (hereafter, i.c.m.) is a partial map $F : 2^{<\omega} \rightarrow \omega$ such that $\sum_{\sigma \in \text{dom}(F)} 2^{-F(\sigma)} \leq 1$ and the set $\{(\sigma, m) : F(\sigma) \leq m\}$ is c.e.

We note that K is an i.c.m., and, as stated before, is identifiable as the *minimal* i.c.m. The salient property of i.c.m.'s is that for every i.c.m. F , $\{(\sigma, m) : F(\sigma) \leq m\}$ is a KC set, and so by the KC theorem and the invariance property of K , there will be a c such that for every $\sigma \in \text{dom}(F)$,

$$K(\sigma) \leq F(\sigma) + c.$$

The information-theoretic study of randomness also involves the notion of compressibility, which is formalized in the following definition.

Definition 2.9. Given a string σ and a $c \in \omega$, σ is called *c-incompressible* if

$$K(\sigma) \geq |\sigma| - c.$$

2.3. Computable measures and Bernoulli measures on 2^ω . We assume that the reader is familiar with basic measure theory. Recall that a measure μ on 2^ω is a *probability measure* if $\mu(2^\omega) = 1$, and μ is a *positive measure* if $\mu(\llbracket \sigma \rrbracket) > 0$ for every string σ .

We now define what it means for a measure on 2^ω to be computable.

Definition 2.10. A measure μ on 2^ω is *computable* if $\sigma \mapsto \mu(\llbracket \sigma \rrbracket)$ is a computable as a real-valued function, i.e. there is a computable function $f : 2^{<\omega} \times \omega \rightarrow \mathbb{Q}_2$ such that

$$|\mu(\llbracket \sigma \rrbracket) - f(\sigma, i)| \leq 2^{-i}$$

for every $\sigma \in 2^{<\omega}$ and $i \in \omega$. The measure μ is *exactly computable* if the function $\sigma \mapsto \mu(\llbracket \sigma \rrbracket)$ can be viewed as a total computable function from $2^{<\omega}$ to \mathbb{Q}_2 .

Hereafter, we denote $\mu(\llbracket \sigma \rrbracket)$ by $\mu(\sigma)$ for strings σ , and $\mu(\llbracket V \rrbracket)$ by $\mu(V)$ for $V \subseteq 2^{<\omega}$. We also denote the *Lebesgue measure* by λ , where $\lambda(\sigma) = 2^{-|\sigma|}$ for every string σ .

Below, we discuss generalized length functions with respect to a special class of measures on Cantor space. Recall that for a probability measure μ on 2^ω and strings $\sigma, \tau \in 2^{<\omega}$, the *conditional probability of $\sigma\tau$ given σ* , denoted $\mu(\sigma\tau \mid \sigma)$, is defined by $\mu(\sigma\tau \mid \sigma) = \mu(\sigma\tau)/\mu(\sigma)$.

Definition 2.11. Given $p \in (0, 1)$, the *Bernoulli p-measure on 2^ω* , denoted μ_p , satisfies $\mu(\sigma 0 \mid \sigma) = p$ for every $\sigma \in 2^{<\omega}$ (so that $\mu(\sigma 1 \mid \sigma) = 1 - p$ for every $\sigma \in 2^{<\omega}$). Thus for each $\sigma \in 2^{<\omega}$ we have

$$\mu_p(\sigma) = p^{\#_0(\sigma)}(1-p)^{\#_1(\sigma)}.$$

Note that λ is the Bernoulli $(1/2)$ -measure, as $\lambda(\sigma) = 2^{-|\sigma|} = (1/2)^{\#_0(\sigma)}(1/2)^{\#_1(\sigma)}$ for $\sigma \in 2^{<\omega}$.

We will now define a measure-theoretic notion of randomness for infinite binary sequences.

Definition 2.12. Let μ be a computable measure on 2^ω .

- (i) A μ -*Martin-Löf test* is a uniformly Σ_1^0 sequence $\{\mathcal{U}_i\}_{i \in \omega}$ such that $\mu(\mathcal{U}_i) \leq 2^{-i}$ for every $i \in \omega$.
- (ii) A sequence $X \in 2^\omega$ *passes* a μ -Martin-Löf test $\{\mathcal{U}_i\}_{i \in \omega}$ if $X \notin \bigcap_{i \in \omega} \mathcal{U}_i$.
- (iii) A sequence $X \in 2^\omega$ is called μ -*Martin-Löf random*, written $X \in \text{MLR}_\mu$, if X passes every μ -Martin-Löf test.

Remark 2.13. For a uniformly Σ_1^0 sequence $\{\mathcal{U}_i\}_{i \in \omega}$ to be a μ -Martin-Löf test, it is actually sufficient that $\mu(\mathcal{U}_i) \leq \alpha_i$ for a computable sequence $\{\alpha_i\}_{i \in \omega}$ of computable reals that converges to 0.

As per the introduction, we aim to generalize the Levin-Schnorr theorem to hold for length functions native to non-Lebesgue measures. That is, we will seek to generalize the following:

Theorem 2.14. A sequence $X \in 2^\omega$ is μ -Martin-Löf random iff

$$(\exists c)(\forall n)[K(X \upharpoonright n) \geq -\log(\mu(X \upharpoonright n)) - c].$$

If $\mu = \lambda$, this is simply

$$(\exists c)(\forall n)[K(X \upharpoonright n) \geq n - c].$$

The following dual concepts measure the density of information in binary sequences.

Definition 2.15. Let $X \in 2^\omega$.

- (i) The *effective Hausdorff dimension* of X is

$$\liminf_{n \rightarrow \infty} \frac{K(X \upharpoonright n)}{n}.$$

- (ii) The *effective packing dimension* of X is

$$\limsup_{n \rightarrow \infty} \frac{K(X \upharpoonright n)}{n}.$$

For Martin-Löf random sequences X , Athreya, Hitchcock, Lutz and Mayordomo [AHLM04] showed that the effective packing dimension of X is 1, and Mayordomo [May02] showed that such sequences also have effective Hausdorff dimension of 1. Hence for any $X \in \text{MLR}$,

$$\liminf_{n \rightarrow \infty} \frac{K(X \upharpoonright n)}{n} = \limsup_{n \rightarrow \infty} \frac{K(X \upharpoonright n)}{n} = \lim_{n \rightarrow \infty} \frac{K(X \upharpoonright n)}{n} = 1.$$

Hoyrup [Hoy12] generalized this result to a wide class of computable measures, namely the shift-invariant computable measures (which we will not define here). In particular, it follows from Hoyrup's result that for every computable Bernoulli measure μ_p (i.e., every Bernoulli measure defined in terms of a computable parameter $p \in (0, 1)$) and every $X \in \text{MLR}_{\mu_p}$,

$$\lim_{n \rightarrow \infty} \frac{K(X \upharpoonright n)}{n} = h(p),$$

where $h(p) = -p \log(p) - (1-p) \log(1-p)$ is the entropy of the measure μ_p .

3. GENERALIZED LENGTH FUNCTIONS AND KOLMOGOROV COMPLEXITY

In this section, we introduce the concept of a generalized length function and a corresponding modification of Kolmogorov complexity.

Definition 3.1. Let $\ell : 2^{<\omega} \rightarrow \omega$ be a function.

- (i) ℓ is called a *generalized length function*, or *g.l.f.*, if ℓ is computable, $\ell(\epsilon) = 0$, and for $\sigma, \tau \in 2^{<\omega}$, if $\sigma \prec \tau$, then $\ell(\sigma) < \ell(\tau)$;
- (ii) ℓ is a *sub-additive g.l.f.* if $\ell(\sigma\tau) \leq \ell(\sigma) + \ell(\tau)$ for every $\sigma, \tau \in 2^{<\omega}$;
- (iii) ℓ is *additive* if equality holds for in the above condition.

Definition 3.2. Let ℓ be a g.l.f. Given a prefix-free machine M and a string $\sigma \in \text{dom}(M)$, define the *prefix-free ℓ -Kolmogorov complexity of σ relative to M* by

$$K_M^{(\ell)}(\sigma) = \{\ell(\tau) : M(\tau) \downarrow = \sigma\}.$$

Given a prefix-free universal machine U , the *ℓ -Kolmogorov complexity of σ* is $K^{(\ell)} = K_U^{(\ell)}$.

If a g.l.f. ℓ is sub-additive, we can prove that some basic properties of K still hold when considering $K^{(\ell)}$. Notably, invariance still holds for $K^{(\ell)}$.

Proposition 3.3 (Invariance). *For any sub-additive g.l.f. ℓ and any prefix-free machine M , for all σ ,*

$$K^{(\ell)}(\sigma) \leq K_M^{(\ell)}(\sigma) + O(1).$$

Proof. This proof follows in the same way as the proof for K . Let M be a prefix-free machine and ρ_M be the coding constant for M . Let σ_M^* be the least string τ in the ℓ -length-lexicographical order (wherein strings are ordered first by ℓ -length and then all strings of the same ℓ -length are ordered by the standard lexicographical order) such that $M(\tau) \downarrow = \sigma$. Then, since $U(\rho_M \hat{\ } \sigma_M^*) = M(\sigma_M^*)$,

$$K^{(\ell)}(\sigma) \leq \ell(\rho_M \hat{\ } \sigma_M^*) \leq \ell(\sigma_M^*) + \ell(\rho_M) = K_M^{(\ell)}(\sigma) + \ell(\rho_M). \quad \square$$

Another basic property of K that also holds for $K^{(\ell)}$ when ℓ is a sub-additive g.l.f. is the following:

Proposition 3.4. *For any additive g.l.f. ℓ and computable function $h : 2^{<\omega} \rightarrow 2^{<\omega}$, for all σ ,*

$$K^{(\ell)}(h(\sigma)) \leq K^{(\ell)}(\sigma) + O(1).$$

Proof. Given ℓ and h as above, let ρ_M be the coding constant of $M = h \circ U$. Let σ^* be the ℓ -length-lexicographically-least string τ such that $U(\tau) \downarrow = \sigma$. We then have $h(U(\sigma^*)) = h(\sigma)$. But then

$$K^{(\ell)}(h(\sigma)) \leq \ell(\rho_M \hat{\ } \sigma^*) \leq \ell(\sigma^*) + \ell(\rho_M) = K^{(\ell)}(\sigma) + \ell(\rho_M). \quad \square$$

3.1. k -length functions. One specific family of g.l.f.'s that we will study here consists of what we call the k -length functions.

Definition 3.5. For $k \geq 1$ and $i \in \{0, 1\}$, let ℓ_k^i be the function defined by

$$\ell_k^i(\sigma) = \#_{1-i}(\sigma) + k\#_i(\sigma)$$

for $\sigma \in 2^{<\omega}$.

Hereafter, we set the convention that $\ell_k = \ell_k^1$ and will refer to $\ell_k(\sigma)$ as the k -length of σ . (All properties of ℓ_k^1 also hold for ℓ_k^0 taking the appropriate symmetries into account.) We will also write $K^{(\ell_k)}(\sigma)$ as $K^{(k)}(\sigma)$ and will refer to this as the k -complexity of σ .

Significantly, there is a class of Bernoulli p -measures that are intimately related to the family of k -length functions. We arrive at this class of measures as follows. For $k \geq 1$, let $f_k(x) = x^k + x - 1$. Since f_k is strictly increasing on $[0, 1]$, $f_k(0) = -1$, and $f_k(1) = 1$, by the intermediate value theorem there is a unique $c \in (0, 1)$ such that $f_k(c) = 0$. Hereafter, let p_k denote this unique c .

For $i \in \{0, 1\}$, let λ_k denote the Bernoulli measure with values on cylinder sets given by

$$\lambda_k(\sigma) = p_k^{\#_0(\sigma)}(1 - p_k)^{\#_1(\sigma)}.$$

For every $k \geq 1$ and $i \in \{0, 1\}$, since $p_k^k = 1 - p_k$, it follows that

$$\lambda_k(\sigma) = p_k^{\#_0(\sigma)}(1 - p_k)^{\#_1(\sigma)} = p_k^{\#_0(\sigma)}p_k^{k \cdot \#_1(\sigma)} = p_k^{\ell_k(\sigma)}. \quad (\dagger)$$

Some calculations yield the following values of p_k :

k	p_k
1	0.5
2	≈ 0.61803
3	≈ 0.68233
4	≈ 0.72449
5	≈ 0.75488
10	≈ 0.83508
20	≈ 0.89389
30	≈ 0.91946
50	≈ 0.94399
100	≈ 0.96658

It is straightforward to verify that the sequence $(p_k)_{k \in \omega}$ is strictly increasing and $\lim_{k \rightarrow \infty} p_k = 1$. To be consistent with the convention that $\lambda(\sigma) = 2^{-|\sigma|}$, we use q_k to denote p_k^{-1} for every $k \geq 1$, so that for every such k , $\lambda_k(\sigma) = q_k^{-\ell_k(\sigma)}$. Note that $q_k \in (1, 2]$ for every k , the sequence $(q_k)_{k \in \omega}$ is strictly decreasing, and $\lim_{k \rightarrow \infty} q_k = 1$.

The measures in the family $\{\lambda_k\}_{k \in \omega}$ are the only Bernoulli measures that satisfy the condition (\dagger) , as shown by the following result.

Proposition 3.6. *Let $p \in (0, 1)$. If $f : 2^{<\omega} \rightarrow \omega$ is a function and $\mu_p(\sigma) = p^{f(\sigma)}$ for all $\sigma \in 2^{<\omega}$, then $\mu_p = \lambda_k$ for some $k \in \mathbb{Z}^+$.*

Proof. First, $\mu_p(0) = p$, hence $f(0) = 1$. Since $1 - p = \mu_p(1) = p^{f(1)}$ and $f(1) \in \omega$, set $k = f(1)$ so that $p^k = 1 - p$ and hence $p = p_k$. Then

$$\mu_p(\sigma) = p^{\#_0(\sigma)}(1 - p)^{\#_1(\sigma)} = p^{\#_0(\sigma)}(p^k)^{\#_1(\sigma)} = p_k^{\#_0(\sigma) + k\#_1(\sigma)} = \lambda_k(\sigma).$$

□

As we will see in the ensuing discussion, the values $(p_k)_{k \in \omega}$ and $(q_k)_{k \in \omega}$ play a central role in the study of the properties of k -length and of $K^{(k)}$ -complexity.

3.2. Some inequalities involving k -length. One aspect of the functions $\{\ell_k\}_{k \in \omega}$ that we may ask about is the number of strings of a certain k -length for a fixed $k \geq 1$. For every $k \geq 1$, let S_k^n denote the set of strings of k -length n . As shown by Theorem 3.9 below, the values $(q_k)_{k \in \omega}$ defined above can be used to provide useful bounds on the sizes of these sets.

Proposition 3.7. *For every $k, n \in \omega$,*

$$S_k^{n+k} = (S_k^{n+k-1})^\wedge 0 \cup (S_k^n)^\wedge 1,$$

where for a set $X \subseteq 2^{<\omega}$ and $i \in \{0, 1\}$, $(X)^\wedge i = \{\sigma i : \sigma \in X\}$. If $n < k$, then $S_k^n = \{0^n\}$.

This is easily proven, and we leave the details to the reader. If we set $s_k^n = |S_k^n|$, we may observe the following.

Corollary 3.8. *For any $k \in \omega$,*

$$s_k^{n+k} = s_k^{n+k-1} + s_k^n.$$

As we see, if we rewrite the indices in the previous proposition we have that for any $n \geq k$, $s_k^n = s_k^{n-1} + s_k^{n-k}$. In particular, for $k = 2$, we have

$$s_2^n = s_2^{n-1} + s_2^{n-2}.$$

Since $s_2^0 = s_2^1 = 1$, the sequence $(s_2^n)_{n \in \omega}$ yields the Fibonacci sequence. More generally, for each $k \geq 2$, the above recursion relation for finding the number of strings of k -length n yields a type of generalized Fibonacci sequence (sometimes referred to Fibonacci p -numbers; see, for instance, [KS09]).

The precise value of s_k^n for a fixed $n, k \in \omega$ is rather difficult to obtain, however. The values of the Fibonacci sequence are explicitly expressed by Binet's formula, where for every $n \in \omega$,

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}},$$

where $\phi = q_2 = 1/p_2$ denotes the golden ratio. Stakhov and Rozin [SR06] found a rather unwieldy sequence of generalized Binet formulas for the Fibonacci p -numbers, but we will make use of the following simpler bounds:

Theorem 3.9. *For any $k \geq 1$ and any $n \geq k$,*

$$q_k^{n-k} \leq s_k^n \leq 2q_k^{n-k}.$$

Proof. Let $k \geq 1$. A simple rearrangement of $p_k^k = 1 - p_k$ shows that for any $k \geq 1$,

$$q_k^{-1} + q_k^{-k} = 1,$$

and so for any $m \in \omega$, we can multiply q_k^{m+1-k} on both sides to get

$$q_k^{m-k} + q_k^{m+1-2k} = q_k^{m+1-k}. \quad (1)$$

We shall now establish the first inequality by induction on n . The base case is straightforward: $s_k^k = 2 \geq 1 = q_k^0$. Let $n \geq k$ and suppose that for all $k \leq i \leq n$, $s_k^i \geq q_k^{i-k}$. Then

$$q_k^{n+1-k} = q_k^{n-k} + q_k^{n+1-2k} \leq s_k^n + s_k^{n+1-k} = s_k^{n+1}.$$

where the first equality follows from (1). This concludes the induction.

We establish the second inequality by induction on n . For the base case, $s_k^k = 2 = 2q_k^{k-k}$. Now, let $n \geq k$ and suppose that the statement holds for all i such that $k \leq i \leq n$. Then

$$s_k^{n+1} = s_k^{n+1-k} + s_k^n \leq 2q_k^{n+1-2k} + 2q_k^{n-k} = 2q_k^{n+1-k}.$$

where the last inequality follow from (1). This concludes the induction. \square

One additional inequality involving s_k^n is the following.

Proposition 3.10. *For all $k \geq 1$ and $n \in \omega$, $s_k^n \leq q_k^n$.*

Proof. Let $k \geq 1$ and $n \in \omega$. We have $s_k^0 = 1 = q_k^0$; for $n < k$, $s_k^n = 1 \leq q_k^n$. For $n \geq k$, $s_k^n \leq 2q_k^{n-k}$ by Theorem 3.9. Note that $1/2 \leq p_k < 1$; indeed, if $p_k < 1/2$, then $p_k^k < (1/2)^k \leq 1/2$, and $1 - p_k > 1/2$, contradicting the defining equality for p_k . Thus we have $0 < p_k^k \leq 1/2$, and we thus have $q_k^k \geq 2$. Hence $2q_k^{-k} \leq 1$. We then immediately see that

$$s_k^n \leq 2q_k^{n-k} = q_k^n \cdot 2q_k^{-k} \leq q_k^n,$$

as desired. \square

3.3. Generalizing the KC Theorem. The bounds on s_k^n established above are particularly useful in generalizing other results involving K to $K^{(k)}$, notably the KC theorem and the minimality of K as an information content measure. Let us consider each of these results in turn.

Theorem 3.11 (*k*-KC Theorem). *Let $\{(r_i, \tau_i)\}_{i \in \omega}$ be a computable sequence of pairs (called requests) with $r_i \in \omega$ and $\tau_i \in 2^{<\omega}$ for every i , such that $\sum_{i \in \omega} q_k^{-r_i} \leq q_k^{-k}$. Then there exists a prefix-free machine M and sequence $\{\sigma_i\}_{i \in \omega}$ of strings with $\ell_k(\sigma_i) = r_i$ such that $\text{dom}(M) = \{\sigma_i : i \in \omega\}$ and $M(\sigma_i) = \tau_i$ for every i . Furthermore, one can obtain an index for M effectively from an index of the sequence of requests.*

Note that the bounded sum condition imposed on the requests differs slightly from the bound in the original KC theorem: instead of requiring $\sum_{i \in \omega} q_k^{-r_i} \leq 1$, we have $\sum_{i \in \omega} q_k^{-r_i} \leq q_k^{-k}$. Following the measure-theoretic proof given in [Nie09], we can always construct effectively a prefix-free machine M and sequence of strings $\{\sigma_i\}_{i \in \omega}$ under the condition that $\sum_{i \in \omega} 2^{-|\sigma_i|} = \sum_{i \in \omega} 2^{-r_i} \leq 1$.

This approach no longer works if we replace length with k -length for $k \geq 2$. For example, for $k = 2$, if we consider a finite set of requests

$$\{(2, \tau_0), (4, \tau_1), (4, \tau_2), (4, \tau_3), (4, \tau_4)\}$$

for some strings $\tau_0, \dots, \tau_4 \in 2^{<\omega}$, we can compute that

$$\sum_{i=0}^4 q_2^{-r_i} = \sum_{i=0}^4 \phi^{-r_i} \leq 1,$$

where, recall, ϕ is the golden mean. However, there is no prefix-free set $\{\sigma_0, \dots, \sigma_4\}$ fulfilling these requests. This is because, as can be easily verified, once we will fulfill the request of a string of 2-length 2, there are only three strings of 2-length 4 available for additional requests.

We resolve the problem of requesting too much measure by ensuring that our request sets never contain S_k^n for any $n \in \omega$. Indeed, for every $k \geq 1$ and $n \in \omega$, by Theorem 3.9 we have

$$\lambda_k(S_k^n) = \frac{|S_k^n|}{q_k^n} \geq \frac{q_k^{n-k}}{q_k^n} = q_k^{-k}, \quad (2)$$

which exceeds the bound on requests in the statement of the k -KC theorem.

Lastly, note that in the case that $k = 1$, the above statement of the k -KC theorem differs from the original KC theorem by imposing the bound $1/2$ on the requests rather than 1. As we will see, using this bound allows us to prove the result with a proof that is considerably simpler than that of the original KC theorem. Indeed, in the standard proofs of the KC theorem, one has to carefully choose each string to fulfill each request (choosing the leftmost available string of the requested length); however, if we use the bound $1/2$, then we are guaranteed that we can choose *any* available string that satisfies a given request (which only comes at a cost of increasing the length of each request in the original KC set by one bit). This follows from the following general lemma.

Lemma 3.12. Suppose that $\{\sigma_0, \dots, \sigma_n\}$ is a prefix-free set such that $r = \sum_{i=0}^n q_k^{-\ell_k(\sigma_i)} \leq q_k^{-k}$. Then for every $j \in \omega$, if

$$q_k^{-j} + r \leq q_k^{-k},$$

then there is a $\tau \in 2^{<\omega}$ such that $\ell_k(\tau) = j$ and $\{\sigma_1, \dots, \sigma_n, \tau\}$ is prefix-free.

Proof. Suppose not, so that there is some $j \in \omega$ such that $q_k^{-j} + r \leq q_k^{-k}$ but for all strings τ , if $\ell_k(\tau) = j$, then there is some $0 \leq t \leq n$ such that $\sigma_t \preceq \tau$. But then since $\llbracket S_k^j \rrbracket \subseteq \bigcup_{i=0}^n \llbracket \sigma_i \rrbracket$, we have

$$r = \sum_{i=0}^n q_k^{-\ell_k(\sigma_i)} = \lambda_k \left(\bigcup_{i=0}^n \llbracket \sigma_i \rrbracket \right) \geq \lambda_k(\llbracket S_k^j \rrbracket) \geq q_k^{-k},$$

where the final inequality is given by (2) above. But then $q_k^{-j} + r \geq q_k^{-j} + q_k^{-k} > q_k^{-k}$, a contradiction. \square

Proof of the k -KC theorem. Given a computable sequence $\{(r_i, \tau_i)\}_{i \in \omega}$ of requests such that $\sum_{i \in \omega} q_k^{-r_i} \leq q_k^{-k}$, we define a sequence $(\sigma_i)_{i \in \omega}$ and a prefix-free machine M by recursion. Let $\sigma_0 = 0^{r_0}$. Given $n \in \omega$, if $\sigma_0, \dots, \sigma_n$ are all defined, search for a string τ such that $\ell_k(\tau) = r_{n+1}$ and $\sigma_i \not\preceq \tau$ for any $1 \leq i \leq n$. This process terminates since such τ exists by Lemma 3.12, and the process is computable since ℓ_k is computable. Set $\sigma_{n+1} = \tau$ and define $M(\sigma_{n+1}) = \tau_{n+1}$. This concludes the construction. \square

Just as with standard complexity, we can define a notion of information content measure for $K^{(k)}$ and generalize the fact that for any information content measure F , there is a c such that for any $\sigma \in \text{dom}(F)$, $K(\sigma) \leq F(\sigma) + 1$. This will be especially useful in giving sufficient conditions for λ_k -Martin-Löf randomness in Section 4.

Definition 3.13. A k -information content measure (hereafter k -i.c.m.) is a partial map $F : 2^{<\omega} \rightarrow \omega$ such that $\sum_{\sigma \in \text{dom}(F)} q_k^{-F(\sigma)} \leq 1$ and the set $\{(\sigma, m) : F(\sigma) \leq m - k + 1\}$ is c.e.

As with traditional i.c.m.'s, the k -complexity $K^{(k)}$ is identifiable as the minimal k -i.c.m., by noting that for every k -i.c.m. F , the set $S = \{(m+1, \sigma) : F(\sigma) \leq m - k + 1\}$ is a k -KC set. This is because of the requirement that the set $\{(\sigma, m) : F(\sigma) \leq m - k + 1\}$ is c.e., and also because, setting $S_1 = \{m+1 \in \omega : \exists \sigma \in 2^{<\omega} (m+1, \sigma) \in S\}$, we have that

$$\begin{aligned} \sum_{m+1 \in S_1} q_k^{-(m+1)} &\leq \sum_{\sigma \in \text{dom}(F)} q_k^{-F(\sigma)-k} \\ &= q_k^{-k} \cdot \sum_{\sigma \in \text{dom}(F)} q_k^{-F(\sigma)} \\ &\leq q_k^{-k}. \end{aligned}$$

We can thus build a prefix-free machine such that for every $\sigma \in \text{dom}(F)$, there is a string τ of k -length $F(\sigma) + k$ such that $M(\tau) = \sigma$. Thus, for every $k \geq 1$ and every k -i.c.m. F , there exists some $c \in \omega$ such that for all $\sigma \in \text{dom}(F)$, $K^{(k)}(\sigma) \leq F(\sigma) + c$.

Using the fact that $K^{(k)}$ is minimal among all k -i.c.m.'s, we can also find an upper bound for the k -complexity of all strings σ in terms of j -length. Here we introduce a multiplicative term $\log q_j / \log q_k$ that we will make use of repeatedly in Sections 4 and 5, which functions as a conversion factor between k -length and j -length.

Theorem 3.14. For any $k \geq 1$, there is a c such that for all σ ,

$$K^{(k)}(\sigma) \leq \frac{\log q_j}{\log q_k} \left(\ell_j(\sigma) + K^{(j)}(\ell_j(\sigma)) \right) + c.$$

Proof. Let $k \geq 1$. It is enough to show that the map $F : 2^{<\omega} \rightarrow \omega$ defined by $F(\sigma) = (\log q_j / \log q_k) (\ell_j(\sigma) + K^{(j)}(\ell_j(\sigma)))$ is a k -i.c.m. Recall that for all $n \in \omega$, $|S_k^n| \leq q_k^n$ by Proposition 3.10. Then since $q_k^{\log q_j / \log q_k} = q_k^{\log_{q_k} q_j} = q_j$, we have

$$\begin{aligned}
\sum_{\sigma \in 2^{<\omega}} q_k^{-\frac{\log q_j}{\log q_k} [\ell_j(\sigma) + K^{(j)}(\ell_j(\sigma))]} &= \sum_{\sigma \in 2^{<\omega}} q_j^{-[\ell_j(\sigma) + K^{(j)}(\ell_j(\sigma))]} \\
&= \sum_{n \in \omega} |S_j^n| q_j^{-(n + K^{(j)}(n))} \\
&\leq \sum_{n \in \omega} q_j^n q_j^{-(n + K^{(j)}(n))} \\
&= \sum_{n \in \omega} q_j^{-K^{(j)}(n)} \\
&\leq \sum_{\sigma \in 2^{<\omega}} q_j^{-K^{(j)}(\sigma)} \\
&\leq \sum_{\tau \in \text{dom}(U)} q_j^{-\ell_j(\tau)} \\
&= \lambda_j(\text{dom}(U)) \\
&\leq 1.
\end{aligned}$$

As in the proof for the upper bound of K , the set $\{(\sigma, m) : F(\sigma) \leq m - k + 1\}$ is c.e. The result then follows by minimality of $K^{(k)}$ among k -i.c.m.'s. \square

Remark 3.15. Note that for $j \leq k$, $q_k \leq q_j$, so $\log q_k \leq \log q_j$. Hence the factor $\log q_j / \log q_k \geq 1$ for $j \leq k$, and, similarly, $\log q_j / \log q_k < 1$ for $j > k$. Approximations of $\log q_j / \log q_k \geq 1$ for $j, k \leq 5$ can be found at the end of Section 5.

4. $K^{(k)}$ -INCOMPRESSIBILITY AND RANDOMNESS

We now consider the notion of $K^{(k)}$ -incompressibility in the context of infinite sequences.

Definition 4.1. For $k, j \geq 1$, a sequence $X \in 2^\omega$ is (k, j) -incompressible if

$$K^{(k)}(X \upharpoonright n) \geq \ell_j(X \upharpoonright n) - O(1).$$

Just as we can characterize Martin-Löf randomness in terms of incompressibility, it is natural to consider whether a similar result holds for (k, j) -incompressibility. First, we have the following result.

Theorem 4.2. For $k, j \geq 1$, if $k < j$, then there are no (k, j) -incompressible sequences.

To show this, we will require the following lemma:

Lemma 4.3. For every $k \geq 1$, there is a c such that for all $n \geq 1$, $K^{(k)}(n) \leq (k + 1) \log n + c$.

Proof. Let $k \geq 1$. For each m , let $\{\sigma_i^m\}_{i \leq 2^m}$ be a strictly-increasing enumeration of 2^m with respect to the lexicographic ordering. We use the prefix-free encoding of the positive integers given by, for every $n \geq 1$, $\rho_n = 0^{\lfloor \log n \rfloor + 1} 1 \sigma_i^{\lfloor \log n \rfloor + 1}$, where $\sigma_i^{\lfloor \log n \rfloor + 1}$ is the binary representation

of n . Given n , we have that

$$\begin{aligned}
\ell_k(\rho_n) &= \ell_k(0^{\lfloor \log n \rfloor + 1} 1 \sigma_i^{\lfloor \log n \rfloor + 1}) \\
&= \ell_k(0^{\lfloor \log n \rfloor + 1}) + \ell_k(1) + \ell_k(\sigma_i^{\lfloor \log n \rfloor + 1}) \\
&= \lfloor \log n \rfloor + 1 + k + \ell_k(\sigma_i^{\lfloor \log n \rfloor + 1}) \\
&\leq \lfloor \log n \rfloor + 1 + k + \ell_k(1^{\lfloor \log n \rfloor + 1}) \\
&= \lfloor \log n \rfloor + 1 + k + k(\lfloor \log n \rfloor + 1) \\
&= (k+1)\lfloor \log n \rfloor + 2k + 1.
\end{aligned}$$

Now, let $c \in \omega$ be such that for all $n \geq 1$, $K^{(k)}(n) \leq \ell_k(\rho_n) + e$. Thus $K^{(k)}(n) \leq (k+1)\lfloor \log n \rfloor + 2k + 1 + e$, and the result follows. \square

Proof of Theorem 4.2. For $k < j$, suppose that $X \in 2^\omega$ is (k, j) -incompressible, so that there is some d such that for all $n \in \omega$,

$$K^{(k)}(X \upharpoonright n) \geq \ell_j(X \upharpoonright n) - d. \quad (3)$$

By Theorem 3.14, for all $n \in \omega$,

$$K^{(k)}(X \upharpoonright n) \leq \frac{\log q_j}{\log q_k} \left(\ell_j(X \upharpoonright n) + K^{(j)}(\ell_j(X \upharpoonright n)) \right) + c. \quad (4)$$

Combining (3) and (4) and rearranging yields

$$\left(1 - \frac{\log q_j}{\log q_k} \right) \ell_j(X \upharpoonright n) \leq \frac{\log q_j}{\log q_k} K^{(j)}(\ell_j(X \upharpoonright n)) + c + d. \quad (5)$$

Applying Lemma 4.3 to (5) yields

$$\left(1 - \frac{\log q_j}{\log q_k} \right) \ell_j(X \upharpoonright n) \leq \frac{\log q_j}{\log q_k} (k+1) \log(\ell_j(X \upharpoonright n)) + c + d$$

or equivalently,

$$\frac{\left(1 - \frac{\log q_j}{\log q_k} \right)}{\frac{\log q_j}{\log q_k} (k+1)} \ell_j(X \upharpoonright n) \leq \log(\ell_j(X \upharpoonright n)) + c + d. \quad (6)$$

By Remark 3.15, since $k < j$, we have $\frac{\log q_j}{\log q_k} < 1$, so setting $r = \frac{\log q_j}{\log q_k}$, (6) becomes

$$\frac{1-r}{r(k+1)} \ell_j(X \upharpoonright n) \leq \log(\ell_j(X \upharpoonright n)) + c + d, \quad (7)$$

where the term on the left-hand side of the inequality is positive. Let N be the least such that for all $n \geq N$,

$$\frac{1-r}{r(k+1)} 2^n > n + c + d;$$

such an N exists by a routine calculation. Since the function $n \mapsto \ell_j(X \upharpoonright n)$ is unbounded, we can find some n such that $\ell_j(X \upharpoonright n) \geq 2^N$, so that

$$\frac{1-r}{r(k+1)} \ell_j(X \upharpoonright n) > \log(\ell_j(X \upharpoonright n)) + c + d,$$

which contradicts (7). Thus, no (k, j) -incompressible sequences exist. \square

Next, we investigate (k, j) -incompressible sequences for $k \leq j$. In particular, we show that there is a connection between a sequence being (k, j) -incompressible and its being complex.

Definition 4.4. $X \in 2^\omega$ is *complex* if there is some computable order (that is, a computable, unbounded, non-decreasing function) $f : \omega \rightarrow \omega$ such that

$$K(X \upharpoonright n) \geq f(n).$$

We now show that for $k \leq j$ all (k, j) -incompressible sequences are complex. We will make use of an effective version of the law of large numbers established by Davie in [Dav01]. Let $\mathcal{K}_c = \{X \in 2^\omega : (\forall n) K(X \upharpoonright n) \geq n - c\}$.

Theorem 4.5 (Davie [Dav01]). *For any $c \in \omega$ and $\epsilon > 0$, we can effectively find $n(c, \epsilon) \in \omega$ such that if $X \in \mathcal{K}_c$ then for all $n > n(c, \epsilon)$,*

$$\left| \frac{\#_0(X \upharpoonright n)}{n} - \frac{1}{2} \right| < \epsilon.$$

Theorem 4.5 does not just apply to infinite sequences but also to sufficiently long incompressible strings. Indeed, for a fixed $c \in \omega$ and $\epsilon > 0$, Theorem 4.5 provides a bound $n(c, \epsilon)$ such that any c -incompressible string σ of length exceeding $n(c, \epsilon)$ satisfies the above condition on $\#_0(\sigma)$. We will apply Theorem 4.5 to a specific collection sufficiently incompressible finite strings.

For each σ , let σ^* be the lexicographically least string such that $U(\sigma^*) = \sigma$.

Lemma 4.6. *For $\epsilon \in (0, 1)$, there is some c and $n(c, \epsilon)$ such that for all strings σ such that $|\sigma^*| \geq n(c, \epsilon)$,*

$$\left| \frac{\#_0(\sigma^*)}{|\sigma^*|} - \frac{1}{2} \right| < \epsilon.$$

Proof. Let c be the coding constant of the machine $U \circ U$. We claim that for sufficiently long strings σ , σ^* is c -incompressible. Indeed, if there is some σ such that $|\sigma^*| > c$ and $|\sigma^*|$ is c -compressible, then there is some τ such that $U(\tau) = \sigma^*$ and $|\tau| < |\sigma^*| - c$. But then $U(U(\tau)) = \sigma$, so that

$$K(\sigma) \leq K_{U \circ U}(\sigma) + c \leq |\tau| + c < |\sigma^*| = K(\sigma),$$

a contradiction. Applying Davie's theorem to any string σ^* with $|\sigma^*| \geq n(c, \epsilon)$ yields the conclusion. \square

Theorem 4.7. *For $k \geq j$, every (k, j) -incompressible sequence is complex.*

Proof. For $k \geq j$, let X be (k, j) -incompressible. Let $d \in \omega$ be such that for every $n \in \omega$, if $U(\tau) = X \upharpoonright n$, then $\ell_k(\tau) \geq \ell_j(X \upharpoonright n) - d$. By definition of ℓ_k and ℓ_j , this yields

$$|\tau| + (k-1) \cdot \#_1(\tau) = \ell_k(\tau) \geq \ell_j(X \upharpoonright n) - d \geq n - d. \quad (8)$$

Fix some rational $\epsilon \in (0, 1)$. For all sufficiently large $n \in \omega$, we have $K(X \upharpoonright n) \geq n(c, \epsilon)$, where c is the coding constant of $U \circ U$. Applying Lemma 4.6 to $\tau = (X \upharpoonright n)^*$ yields

$$\left| \frac{\#_0(\tau)}{|\tau|} - \frac{1}{2} \right| < \epsilon,$$

which implies that

$$\#_1(\tau) \leq (1/2 + \epsilon)|\tau|. \quad (9)$$

From (9) it follows that

$$|\tau| + (k-1) \cdot \#_1(\tau) \leq |\tau| + (k-1)(1/2 + \epsilon)|\tau|.$$

Combining this inequality with (8) and the fact that $|\tau| = K(X \upharpoonright n)$ yields

$$K(X \upharpoonright n)((k-1)(1/2 + \epsilon)) \geq n - d,$$

from which it follows that

$$K(X \upharpoonright n) \geq \frac{n}{(k-1)(1/2 + \epsilon)} - e$$

for some $e \in \omega$. Setting

$$f(n) = \left\lfloor \frac{n}{(k-1)(1/2 + \epsilon)} \right\rfloor - e$$

yields the desired computable order that witnesses that X is complex. \square

As we will show shortly, (k, j) -incompressibility does not guarantee randomness, at least for $j = 1$: there are $(k, 1)$ -incompressible sequences that are not random with respect to any computable measure. However, if we modify the definition of (k, j) -incompressibility for any k and j (even when $k < j$), we do get a characterization of randomness.

Definition 4.8. For $k, j \geq 1$, a sequence $X \in 2^\omega$ is *generalized (k, j) -incompressible* if

$$K^{(k)}(X \upharpoonright n) \geq \frac{\log q_j}{\log q_k} \ell_j(X \upharpoonright n) - O(1).$$

Remark 4.9. Using the fact that $(\log p_j)/(\log p_k) = (\log q_j)/(\log q_k)$, we can equivalently define X to be generalized (k, j) -incompressible if

$$K^{(k)}(X \upharpoonright n) \geq \frac{\log p_j}{\log p_k} \ell_j(X \upharpoonright n) - O(1).$$

Theorem 4.10. For any $X \in 2^\omega$ and $j, k \geq 1$, $X \in MLR_{\lambda_j}$ if and only if X is generalized (k, j) -incompressible.

To prove Theorem 4.10, we need two lemmas.

Lemma 4.11. For every $k \geq 1$,

$$\sum_{n \in \omega} q_k^{-(kn+1)} \leq 1.$$

Proof. It suffices to show that for every $n \in \omega$,

$$\sum_{i=0}^n q_k^{-(ki+1)} + q_k^{-(n+1)k} = 1.$$

Once we have established this equality, it will follow that for every $n \in \omega$,

$$\sum_{i=0}^n q_k^{-(ki+1)} < \sum_{i=0}^n q_k^{-(ki+1)} + q_k^{-(n+1)k} = 1,$$

and thus the sequence $\{\sum_{i=0}^n q_k^{-(ki+1)}\}_{n \in \omega}$ will be (strictly) increasing and bounded above by 1. Therefore the limit of this sequence must exist and is at most 1.

To establish the equality, we proceed by induction on n . For the base case, this is just

$$q_k^{-1} + q_k^{-k} = 1.$$

Now let $n \in \omega$ and suppose that the statement holds for n . Then

$$\begin{aligned} \sum_{i=0}^{n+1} q_k^{-(ki+1)} + q_k^{-(n+2)k} &= \sum_{i=0}^n q_k^{-(ki+1)} + q_k^{-[k(n+1)+1]} + q_k^{-(n+2)k} \\ &= \sum_{i=0}^n q_k^{-(ki+1)} + q_k^{-(n+1)k} = 1. \end{aligned}$$

This concludes the induction. □

Lemma 4.12. Let M be a prefix-free machine, let $n \in \omega$, $j, k \geq 1$, and let

$$S_n = \left\{ \sigma : K_M^{(k)}(\sigma) < \frac{\log q_j}{\log q_k} (\ell_j(\sigma) - n) \right\}.$$

Then $\lambda_j(S_n) \leq q_j^{-n} \lambda_k(\text{dom}(M))$.

Proof. For every $\sigma \in S_n$, let $\tau_\sigma \in \text{dom}(M)$ be such that $M(\tau_\sigma) \downarrow = \sigma$ and

$$\ell_k(\tau_\sigma) < \frac{\log q_j}{\log q_k} (\ell_j(\sigma) - n).$$

Noting, as in the proof of Theorem 3.14, that $q_j^{\log q_k / \log q_j} = q_j^{\log_{q_j} q_k} = q_k$, we have

$$\begin{aligned}
\lambda_j(S_n) &\leq \sum_{\sigma \in S_n} q_j^{-\ell_j(\sigma)} \\
&< \sum_{\sigma \in S_n} q_j^{-\frac{\log q_k}{\log q_j} \ell_k(\tau_\sigma) - n} \\
&= q_j^{-n} \sum_{\sigma \in S_n} q_k^{-\ell_k(\tau_\sigma)} \\
&\leq q_j^{-n} \sum_{\tau \in \text{dom}(M)} q_k^{-\ell_k(\tau)} \\
&= q_j^{-n} \lambda_k(\text{dom}(M)),
\end{aligned}$$

as desired. □

Proof of Theorem 4.10. (\rightarrow): Let $X \in 2^\omega$ and suppose that $X \in \text{MLR}_{\lambda_j}$. For any n , let

$$R_n = \left\{ \sigma \in 2^{<\omega} : K^{(k)}(\sigma) < \frac{\log q_j}{\log q_k} (\ell_j(\sigma) - n) \right\}.$$

By Lemma 4.12, the sequence $\{\mathcal{U}_n\}_{n \in \omega}$ such that $\mathcal{U}_n = \llbracket R_n \rrbracket$ for every n is uniformly Σ_1^0 and $\lambda_j(\mathcal{U}_n) \leq q_j^{-n}$ for every $n \in \omega$. Since $\{q_j^{-n}\}_{n \in \omega}$ is a computable sequence of computable reals, by Remark 2.13 it follows that $\{\mathcal{U}_n\}_{n \in \omega}$ is a λ_j -Martin-Löf test, so $X \notin \bigcap_{n \in \omega} \mathcal{U}_n$. Thus there is a c such that for all n ,

$$K^{(k)}(X \upharpoonright n) \geq \frac{\log q_j}{\log q_k} (\ell_j(X \upharpoonright n) - c).$$

(\leftarrow): Suppose $X \notin \text{MLR}_{\lambda_j}$. Let $\{\mathcal{U}_n\}_{n \in \omega}$ be a λ_j -Martin-Löf test that X fails, and let $\{R_n\}_{n \in \omega}$ be a uniformly c.e. sequence of prefix-free sets such that $\mathcal{U}_n = \llbracket R_n \rrbracket$. Let F be defined on $\bigcup_{n \geq 1} R_{n(j+1)+1}$ so that for $\sigma \in R_{n(j+1)+1}$ (where this is the greatest such n),

$$F(\sigma) = \left\lceil \frac{\log q_j}{\log q_k} (\ell_j(\sigma) - n) \right\rceil.$$

We verify that F is a k -i.c.m. Again using the fact that $q_k^{\log q_j / \log q_k} = q_j$, we have

$$\begin{aligned}
\sum_{n \geq 1} \sum_{\sigma \in R_{n(j+1)+1}} q_k^{-F(\sigma)} &\leq \sum_{n \geq 1} \sum_{\sigma \in R_{n(j+1)+1}} q_k^{-\frac{\log q_j}{\log q_k}(\ell_j(\sigma)-n)} \\
&= \sum_{n \geq 1} \sum_{\sigma \in R_{n(j+1)+1}} q_j^{-(\ell_j(\sigma)-n)} \\
&= \sum_{n \geq 1} q_j^n \sum_{\sigma \in R_{n(j+1)+1}} q_j^{-\ell_j(\sigma)} \\
&= \sum_{n \geq 1} q_j^n \lambda_j(\llbracket R_{n(j+1)+1} \rrbracket) \\
&\leq \sum_{n \geq 1} q_j^n 2^{-n(j+1)-1} \\
&\leq \sum_{n \geq 1} q_j^n q_j^{-n(j+1)-1} && (\text{since } q_j \leq 2) \\
&= \sum_{n \geq 1} q_j^{-(jn+1)} \leq 1. && (\text{by Lemma 4.11})
\end{aligned}$$

Next, the set $\{(\sigma, m) : F(\sigma) \leq m - k + 1\}$ is clearly c.e., as each of the values $\log q_j$ and $\log q_k$ is either equal to 1 or is irrational. By the minimality of $K^{(k)}$ among all k -i.c.m.s, there is some $d \in \omega$ such that for all σ , if $\sigma \in R_{n(j+1)+1}$ for some $n \geq 1$, then $K^{(k)}(\sigma) \leq F(\sigma) + d$. Since $X \in \bigcap_{n \in \omega} \mathcal{U}_n$, it follows that for every $n \geq 1$, $X \in \mathcal{U}_{n(j+1)+1}$. For each $n \geq 1$, let \hat{n} be the least integer such that

$$\frac{\log q_j}{\log q_k} n + d + 1 \leq \frac{\log q_j}{\log q_k} \hat{n}.$$

Then since $X \in \mathcal{U}_{\hat{n}(j+1)+1}$, there is some m such that $X \upharpoonright m \in R_{\hat{n}(j+1)+1}$, so that

$$\begin{aligned}
K^{(k)}(X \upharpoonright m) &\leq F(X \upharpoonright m) + d \\
&\leq \frac{\log q_j}{\log q_k}(\ell_j(X \upharpoonright m) - \hat{n}) + d + 1 \\
&\leq \frac{\log q_j}{\log q_k}(\ell_j(X \upharpoonright m) - n),
\end{aligned}$$

which yields the conclusion. \square

One consequence of Theorem 4.10 is a characterization of (k, j) -incompressibility when $k = j$.

Corollary 4.13. *For any $X \in 2^\omega$ and $k \geq 1$, $X \in \text{MLR}_{\lambda_k}$ if and only if for all n ,*

$$K^{(k)}(X \upharpoonright n) \geq \ell_k(X \upharpoonright n) - O(1).$$

We now conclude this section by showing that there are sequences that are $(k, 1)$ -incompressible and yet not random with respect to any computable measure. To do so, we use the following generalization of the above-discussed theorem of Davie's, Theorem 4.5, which is an immediate consequence of [HR09, Theorem 5.2.3] due to Hoyrup and Rojas. For $p \in (0, 1)$, let $\mathcal{K}_c^p = \{X \in 2^\omega : (\forall n) K(X \upharpoonright n) \geq -\log \mu_p(X \upharpoonright n) - c\}$, where μ_p is the Bernoulli p -measure.

Theorem 4.14. *For any $c \in \omega$ and $\epsilon > 0$, we can effectively find $n(c, \epsilon) \in \omega$ such that if $X \in \mathcal{K}_c^p$ then for all $n > n(c, \epsilon)$,*

$$\left| \frac{\#0(X \upharpoonright n)}{n} - p \right| < \epsilon.$$

The proof of Theorem 4.14 also uses the fact that the sequence $(\mathcal{K}_c^p)_{c \in \omega}$ defines a universal μ_p -Martin-Löf test. That this fact can be used in tandem with [HR09, Theorem 5.2.3] is noted in Section 5.2.4 of [HR09]. As there are no new ideas involved in the proof, we leave the details to the reader.

Just as Theorem 4.5 applies to sufficiently long incompressible strings, so too does Theorem 4.14 apply to sufficiently long k -incompressible strings. We first need an auxiliary lemma.

Lemma 4.15. *For each $k \in \omega$, there is some computable function f such that for every $e \in \omega$ and every $\sigma \in 2^{<\omega}$, if*

$$K^{(k)}(\sigma) \geq \ell_k(\sigma) - e$$

then

$$K(\sigma) \geq -\log \lambda_k(\sigma) - f(e).$$

Proof. Note that the collection $\{R_n\}_{n \in \omega}$ defined by

$$R_n = \{\sigma \in 2^{<\omega} : K(\sigma) < -\log \lambda_k(\sigma) - n\}$$

defines a universal λ_k -Martin-Löf test by the Levin-Schnorr theorem. Applying the (\leftarrow) direction of the proof of Theorem 4.10 to $\{R_n\}_{n \in \omega}$ yields a $d \in \omega$ such that for every $\sigma \in R_{n(k+1)+1}$,

$$K^{(k)}(\sigma) \leq \ell_k(\sigma) - n + d.$$

Taking the converse, we have for each $e \in \omega$ and every σ ,

$$K^{(k)}(\sigma) \geq \ell_k(\sigma) - e \text{ implies } K(\sigma) \geq -\log \lambda_k(\sigma) - ((e + d)(k + 1) + 1).$$

Setting $f(e) = (e + d)(k + 1) + 1$ yields the desired function. \square

We now state and prove the finitary version of Theorem 4.14. To do so, we need to generalize the notion of a “shortest description” to k -complexity. For each σ , let $\sigma^{*(k)}$ be the length-lexicographically least string such that $K^{(k)}(\sigma) = \ell_k(\sigma^{*(k)})$.

Lemma 4.16. *For $k \in \omega$ and $\epsilon \in (0, 1)$, there is some d and $n(d, \epsilon)$ such that for all strings σ such that $|\sigma^{*(k)}| \geq n(d, \epsilon)$,*

$$\left| \frac{\#_0(\sigma^{*(k)})}{|\sigma^{*(k)}|} - p_k \right| < \epsilon.$$

Proof. Fix k and ϵ as above. As in the proof of Lemma 4.6, let c be the coding constant of $U \circ U$. One can easily verify that for each $\sigma \in 2^{<\omega}$, $K^{(k)}(\sigma^{*(k)}) \geq \ell_k(\sigma^{*(k)}) - c$. By Lemma 4.15, it follows that $K(\sigma) \geq -\log \lambda_k(\sigma) - f(c)$. Setting $d = f(c)$ and applying Theorem 4.14 to any string σ such that $|\sigma^{*(k)}| \geq n(d, \epsilon)$ yields the desired conclusion. \square

Theorem 4.17. *For $k \in \omega$ and $\epsilon \in (0, 1 - p_k)$, if for all n ,*

$$K(X \upharpoonright n) \geq \frac{n}{1 + (k - 1)(1 - p_k - \epsilon)} - O(1),$$

then X is $(k, 1)$ -incompressible.

Proof. Given X as in the hypothesis, let d and $n(d, \epsilon)$ be as in Lemma 4.16. Then for all n such that $K^{(k)}(X \upharpoonright n) \geq n(d, \epsilon)$ and for all τ such that $U(\tau) = X \upharpoonright n$, we have

$$|\tau| \geq \frac{n}{1 + (k - 1)(1 - p_k - \epsilon)} - O(1). \quad (10)$$

In particular, if $\tau = (X \upharpoonright n)^{*(k)}$, then by Lemma 4.16, we have

$$\left| \frac{\#_0(\tau)}{|\tau|} - p_k \right| < \epsilon.$$

This implies that

$$\left| \frac{\#_1(\tau)}{|\tau|} - (1 - p_k) \right| < \epsilon,$$

and hence that

$$\#_1(\tau) \geq (1 - p_k - \epsilon)|\tau|. \quad (11)$$

By the choice of τ , we have

$$K^{(k)}(X \upharpoonright n) = |\tau| + (k-1)\#_1(\tau),$$

which when combined with (11) yields

$$K^{(k)}(X \upharpoonright n) \geq (1 + (k-1)(1 - p_k - \epsilon))|\tau|. \quad (12)$$

Combining (10) and (12) gives

$$K^{(k)}(X \upharpoonright n) \geq n - O(1).$$

□

Corollary 4.18. *For $k > 1$, there is a $(k, 1)$ -incompressible sequence that is not Martin-Löf random with respect to any computable measure.*

Proof. Choose ϵ such that $p_k + \epsilon < 1$ and let $\delta > 0$ satisfy $\delta \leq (k-1)(1 - p_k - \epsilon)$. By Miller [Mil11], for every $\alpha \in (0, 1)$, there is some $X \in 2^\omega$ such that (i) $K(X \upharpoonright n) \geq \alpha n - O(1)$ and (ii) X does not compute any sequence Y satisfying $K(Y \upharpoonright n) \geq \beta n - O(1)$ for some $\beta > \alpha$ in $(0, 1]$. Let $\alpha = \frac{1}{1+\delta}$, and let X satisfy (i) and (ii) for this choice of X . First, we have

$$K(X \upharpoonright n) \geq \alpha n - O(1) = \frac{n}{1+\delta} - O(1) \geq \frac{n}{1 + (k-1)(1 - p_k - \epsilon)},$$

and thus by Theorem 4.17, X is $(k, 1)$ -incompressible. Next, it follows from (ii) that X does not compute a Martin-Löf random sequence. Since every sequence that is random with respect to some computable measure must compute a Martin-Löf random sequence (by a result due independently to Zvonkin/Levin [ZL70] and Kautz [Kau91]), it follows that X is not random with respect to any computable measure.

□

Our analysis of (k, j) -incompressibility is not complete, as the following is still open.

Question 4.19. *For $j > 1$ and $k > j$, is there a (k, j) -incompressible sequence that is not Martin-Löf random with respect to any computable measure?*

5. EFFECTIVE DIMENSION AND GENERALIZED LENGTH FUNCTIONS

In this final section we consider effective Hausdorff dimension, effective packing dimension, and entropy in the context of generalized length functions. As stated at the end of Section 2, it follows from a result of Hoyrup [Hoy12] that for each computable Bernoulli measure μ_p and each $X \in \text{MLR}_{\mu_p}$,

$$\lim_{n \rightarrow \infty} \frac{K(X \upharpoonright n)}{n} = -p \log(p) - (1-p) \log(1-p) = h(p)$$

However, if we consider modified effective k -packing and k -Hausdorff dimensions (for $k \geq 1$), defined similarly as

$$\limsup_{n \rightarrow \infty} \frac{K^{(k)}(X \upharpoonright n)}{n} \text{ and } \liminf_{n \rightarrow \infty} \frac{K^{(k)}(X \upharpoonright n)}{n},$$

respectively, we will see that for any $j \geq 1$, if $X \in \text{MLR}_{\lambda_j}$, the effective k -packing and effective k -Hausdorff dimensions will again coincide and will be equal to $h(p_j)$ multiplied by the conversion factor $-1/\log p_k$.

Theorem 5.1. *For every $j, k \geq 1$, and all $X \in \text{MLR}_{\lambda_j}$,*

$$\lim_{n \rightarrow \infty} \frac{K^{(k)}(X \upharpoonright n)}{n} = -\frac{1}{\log p_k} h(p_j).$$

Proof of Theorem 5.1. Let $j, k \geq 1$. First, we will use the well-known fact that law of large numbers holds for all sequences that are random with respect to a Bernoulli measure; that is, given a computable real $p \in (0, 1)$ and the Bernoulli p -measure μ_p , if $X \in \text{MLR}_{\mu_p}$ then

$$\lim_{n \rightarrow \infty} \frac{\#_0(X \upharpoonright n)}{n} = p \text{ and } \lim_{n \rightarrow \infty} \frac{\#_1(X \upharpoonright n)}{n} = 1 - p.$$

With this in mind, let $X \in \text{MLR}_{\lambda_j}$. We first show that the effective k -packing and effective k -Hausdorff dimensions of X are equal. We handle effective k -packing dimension first. By Theorem 4.10, since $X \in \text{MLR}_{\lambda_j}$, there is some $c \in \omega$ such that for every $n \in \omega$,

$$K^{(k)}(X \upharpoonright n) \geq \frac{\log p_j}{\log p_k} (\ell_j(X \upharpoonright n) - c).$$

Hence for every $n \in \omega$,

$$\frac{K^{(k)}(X \upharpoonright n)}{n} \geq \frac{\log p_j}{\log p_k} \left(\frac{\ell_j(X \upharpoonright n) - c}{n} \right)$$

and thus

$$\liminf_{n \rightarrow \infty} \frac{K^{(k)}(X \upharpoonright n)}{n} \geq \liminf_{n \rightarrow \infty} \frac{\log p_j}{\log p_k} \left(\frac{\ell_j(X \upharpoonright n) - c}{n} \right) = \lim_{n \rightarrow \infty} \frac{\log p_j}{\log p_k} \left(\frac{\ell_j(X \upharpoonright n) - c}{n} \right),$$

assuming the latter limit exists. We verify that it does as follows:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\log p_j}{\log p_k} \left(\frac{\ell_j(X \upharpoonright n) - c}{n} \right) &= \frac{\log p_j}{\log p_k} \cdot \lim_{n \rightarrow \infty} \frac{\#_0(X \upharpoonright n) + j\#_1(X \upharpoonright n) - c}{n} \\ &= \frac{\log p_j}{\log p_k} (p_j + j(1 - p_j)), \end{aligned}$$

where the latter equality follows from the law of large numbers. Next, by Theorem 3.14, let $c' \in \omega$ be such that for every $n \in \omega$,

$$K^{(k)}(X \upharpoonright n) \leq \frac{\log p_j}{\log p_k} (\ell_j(X \upharpoonright n) + K^{(j)}(\ell_j(X \upharpoonright n))) + c'. \quad (13)$$

Lemma 4.3 gives us that there is some $c'' \in \omega$ such that for every $n \in \omega$,

$$K^{(j)}(\ell_j(X \upharpoonright n)) \leq (j + 1) \log \ell_j(X \upharpoonright n) + c''. \quad (14)$$

Lastly, for all $n \in \omega$,

$$\ell_j(X \upharpoonright n) \leq jn, \quad (15)$$

since we can have at most n 1's in any string of length n . Thus we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{K^{(k)}(X \upharpoonright n)}{n} &\leq \limsup_{n \rightarrow \infty} \left[\frac{\log p_j}{\log p_k} \left(\frac{\ell_j(X \upharpoonright n) + K^{(j)}(\ell_j(X \upharpoonright n))}{n} \right) + \frac{c'}{n} \right] && \text{by (13)} \\ &\leq \limsup_{n \rightarrow \infty} \left[\frac{\log p_j}{\log p_k} \left(\frac{\ell_j(X \upharpoonright n) + (j + 1) \log \ell_j(X \upharpoonright n) + c''}{n} \right) + \frac{c'}{n} \right] && \text{by (14)} \\ &\leq \limsup_{n \rightarrow \infty} \left[\frac{\log p_j}{\log p_k} \left(\frac{\ell_j(X \upharpoonright n) + (j + 1) \log jn + c''}{n} \right) + \frac{c'}{n} \right] && \text{by (15)} \\ &= \lim_{n \rightarrow \infty} \left[\frac{\log p_j}{\log p_k} \left(\frac{\ell_j(X \upharpoonright n) + (j + 1) \log jn + c''}{n} \right) + \frac{c'}{n} \right], \end{aligned}$$

again assuming that the latter limit exists. Now, following similar steps as those taken with the first limit, we see that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left[\frac{\log p_j}{\log p_k} \left(\frac{\ell_j(X \upharpoonright n) + (j+1) \log jn + c''}{n} \right) + \frac{c'}{n} \right] \\
&= \frac{\log p_j}{\log p_k} \left[\lim_{n \rightarrow \infty} \frac{\ell_j(X \upharpoonright n)}{n} + \lim_{n \rightarrow \infty} \frac{(j+1) \log jn}{n} + \lim_{n \rightarrow \infty} \frac{c''}{n} \right] + \lim_{n \rightarrow \infty} \frac{c'}{n} \\
&= \frac{\log p_j}{\log p_k} (p_j + j(1 - p_j) + 0 + 0) + 0 \\
&= \frac{\log p_j}{\log p_k} (p_j + j(1 - p_j)).
\end{aligned}$$

It follows that

$$\limsup_{n \rightarrow \infty} \frac{K^{(k)}(X \upharpoonright n)}{n} = \liminf_{n \rightarrow \infty} \frac{K^{(k)}(X \upharpoonright n)}{n} = \lim_{n \rightarrow \infty} \frac{K^{(k)}(X \upharpoonright n)}{n} = \frac{\log p_j}{\log p_k} (p_j + j(1 - p_j)),$$

as desired. It now remains to show that

$$\lim_{n \rightarrow \infty} \frac{K^{(k)}(X \upharpoonright n)}{n} = -\frac{1}{\log p_k} h(p_j).$$

This is straightforward:

$$\begin{aligned}
h(p_j) &= -p_j \log p_j - (1 - p_j) \log(1 - p_j) \\
&= -p_j \log p_j - (1 - p_j) \log p_j^j \\
&= -p_j \log p_j - j(1 - p_j) \log p_j \\
&= -(\log p_j)(p_j + j(1 - p_j)).
\end{aligned}$$

The full result immediately follows, and this concludes the proof. \square

We conclude with a table of values of $\log p_j / \log p_k$ for $j, k \leq 5$. Note that for $j = 1$, the various values of the effective k -dimension of Martin-Löf random sequences for $k \geq 2$ exceeds 1. In fact, these values grow without bound as $k \rightarrow \infty$, since $p_k \rightarrow 1$.

$j \backslash k$	1	2	3	4	5
1	1	1.4404	1.8133	2.1506	2.4009
2	.69421	1	1.2588	1.493	1.6667
3	.55149	.7944	1	1.1861	1.3241
4	.46498	.6698	.84313	1	1.1164
5	.41651	.5999	.7552	.8958	1

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